Fundamental transformations for quadrilateral lattices: first potentials and τ-functions, symmetric and pseudo-Egorov reductions

2001 J. Phys. A: Math. Gen. 34 10413

(http://iopscience.iop.org/0305-4470/34/48/307)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 10413–10421 PII: S0305-4470(01)23941-9

Fundamental transformations for quadrilateral lattices: first potentials and τ-functions, symmetric and pseudo-Egorov reductions

Manuel Mañas

Departamento de Física Teórica II, Universidad Complutense, E28040 Madrid, Spain

E-mail: manuel@darboux.fis.ucm.es

Received 17 April 2001, in final form 3 September 2001
Published 23 November 2001
Online at stacks.iop.org/JPhysA/34/10413

Abstract
We find the behaviour of first potentials and τ-functions for quadrilateral lattices under vectorial fundamental transformations. We also give those transformations which preserve the symmetric and pseudo-Egorov reductions.

PACS numbers: 05.50.+q, 02.30.-f

1. Introduction

Integrable discrete equations seems to be essential in the understanding of integrable systems. On the one hand many integrable nonlinear PDEs are continuous limits of integrable P∆Es (partial difference equations). On the other hand many of these integrable nonlinear PDEs are reductions or connected with differential geometry, in particular with the theory of conjugate nets and its reductions. Sometime ago the German geometer Sauer discussed for example discrete pseudo-spherical surfaces. Recently, the group of Bobenko and Pinkall and the group of Doliwa and Santini have given to the theory of integrable lattices an almost closed form. These integrable lattices contain as reductions many of the mentioned discrete integrable systems and constitute a cornerstone in the theory of integrable systems. Among these lattices one finds the quadrilateral, circular, Egorov and asymptotic lattices.

In this paper we investigate how the first potentials and τ-functions for quadrilateral lattices transform under fundamental transformations. Using this information we search for those fundamental transformations which reduce to symmetric lattices and pseudo-Egorov lattices. The layout of the paper is as follows: in section 2 we review some well known basic aspects of quadrilateral lattices, reductions and transformations. In section 3 we present the transformation of both first potentials and τ-functions under vectorial fundamental transformations. Finally, section 4 is devoted to characterizing those vectorial fundamental transformations preserving symmetric and pseudo-Egorov lattices.

2. Quadrilateral lattices

Among the N-dimensional lattices $\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{R}^N$ there is a distinguished class for which the elementary quadrilaterals are planar [7,8,15]: the so-called quadrilateral lattice. The planarity condition can be expressed by the linear equation for suitably renormalized tangent vectors $\mathfrak{C}_i(\boldsymbol{n}) \in \mathbb{R}^N$

$$\Delta_j \mathfrak{C}_i = (T_j Q_{ij})\mathfrak{C}_j \qquad i, j = 1, \ldots, N \quad i \neq j \tag{1}$$

with its compatibility conditions being the following discrete Darboux equations [2]:

$$\Delta_k Q_{ij} = (T_k Q_{ik}) Q_{kj} \qquad i, j \text{ and } k \text{ different.}$$

Here we are using the notation $\Delta_j := T_j - 1$ where $T_j f(n_1, \ldots, n_N) = f(n_1, \ldots, n_{j-1}, n_j + 1, n_{j+1}, \ldots, n_N)$. The points $\boldsymbol{x}$ of the lattice satisfy

$$\Delta_i \boldsymbol{x} = (T_i H_i)\mathfrak{C}_i \qquad i = 1, \ldots, N$$

where the H_i fulfil

$$\Delta_i H_j = Q_{ij} T_i H_i \qquad i, j = 1, \ldots, N \quad i \neq j. \tag{2}$$

In the above formulae, T_i is the translation operator in the discrete variable n_i and $\Delta_i = T_i - 1$ is the corresponding partial difference operator.

2.1. *Backward representation, first potentials and τ-functions*

As was explained in [9] there is an equivalent description in terms of backward geometrical objects, $\tilde{\mathfrak{C}}_i$, $\tilde{H}_i$, $\tilde{Q}_{ij}$ which satisfy

$$\Delta_i \tilde{\mathfrak{C}}_j = \tilde{Q}_{ij} T_i \tilde{\mathfrak{C}}_i \qquad \Delta_j \tilde{H}_i = (T_j \tilde{Q}_{ij})\tilde{H}_j.$$

There exist first potentials ρ_i, $i = 1, \ldots, N$, [9] such that

$$\mathfrak{C}_i = -\rho_i T_i \tilde{\mathfrak{C}}_i \qquad T_i H_i = -\frac{1}{\rho_i}\tilde{H}_i.$$

Moreover, as was proven in [9] we have

$$\rho_j T_j \tilde{Q}_{ij} = \rho_i T_i Q_{ji}$$

where the first potentials ρ_i are characterized by

$$\frac{T_j \rho_i}{\rho_i} = 1 - (T_i Q_{ji})(T_j Q_{ij}). \tag{3}$$

The above equation ensures, as was noticed in [9], the existence of a second potential τ such that

$$\rho_i = \frac{T_i \tau}{\tau}$$

which essentially is the τ-function of the quadrilateral lattice as was discussed in [5].

2.2. *Reduced lattices*

Quadrilateral lattices $\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{R}^N$ for which each quadrilateral is inscribed in a circle are called circular or cyclid lattices [1,3,6,10,14]. It can be shown that the constraint

$$\mathfrak{C}_i \cdot T_i(\mathfrak{C}_j) + \mathfrak{C}_j \cdot T_j(\mathfrak{C}_i) = 0 \qquad i \neq j \tag{4}$$

for the tangent vectors is the equivalent circularity property. The first potentials for the circular lattices satisfy $\rho_i = \|\mathfrak{C}_i\|^2$ [9]. In [9] the symmetric and Egorov lattices were introduced—the

Egorov lattice was also considered by Wolfang Schief. The symmetric lattice appears when backward and forward rotation coefficients are the same, which can be cast in the condition

$$(T_i Q_{ji})(T_j Q_{kj})(T_k Q_{ik}) = (T_j Q_{ij})(T_i Q_{ki})(T_k Q_{jk}) \qquad i, j \text{ and } k \text{ different.}$$

This symmetric case is also characterized by the following relation among first potentials and rotation coefficients:

$$\rho_j T_j Q_{ij} = \rho_i T_i Q_{ji}. \tag{5}$$

A circular, symmetric and diagonal invariant lattice is called an Egorov lattice; it was proven that Egorov lattices are characterized by

$$\mathfrak{C}_i \cdot T_i(\mathfrak{C}_j) = 0 \qquad i \neq j. \tag{6}$$

Finally, in [12] pseudo-circular and pseudo-Egorov lattices in pseudo-Euclidean space $R_{p,q}$, $p+q=N$, were introduced. Here, we have a non-degenerate symmetric bilinear form

$$\boldsymbol{X}\cdot\tilde{\boldsymbol{X}} := \sum_{i=1}^{N} \epsilon_i X_i \tilde{X}_i \qquad \text{with} \quad \epsilon_i := \begin{cases} 1 & i = 1, \dots, p \\ -1 & i = p+1, \dots, p+q \end{cases}$$

which can be written as

$$\boldsymbol{X}\cdot\tilde{\boldsymbol{X}} = (X_1, \dots, X_N) I_{p,q} \begin{pmatrix} \tilde{X}_1 \\ \vdots \\ \tilde{X}_N \end{pmatrix}$$

with

$$I_{p,q} := \operatorname{diag}(\epsilon_1, \dots, \epsilon_N).$$

The pseudo-circular and pseudo-Egorov lattices are defined as in (4) and (6) but replacing the Euclidean scalar product by the pseudo-Euclidean scalar product has been introduced. As was pointed out to me by Adam Doliwa during this SIDE IV conference, these pseudo-circular lattices should correspond, through a convenient stereographic projection, to lattices in quadric surfaces as in [4].

2.3. *The discrete fundamental transformation*

The fundamental transformation of Jonas for conjugate nets was *discretized* to quadrilateral lattices by Doliwa, Mañas and Santini [7, 13]. The superposition of a number of fundamental transformations can be compactly formulated in the vectorial fundamental transformation which in turn has a nice geometrical interpretation for quadrilateral lattices [7].

The discrete vectorial fundamental transformation [7, 13] is given by

$$\begin{aligned}
Q'_{ij} &= Q_{ij} - \boldsymbol{\Phi}^*_j \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \boldsymbol{\Phi}_i && i, j = 1, \dots, N \quad i \neq j \\
H'_i &= H_i - \boldsymbol{\Phi}^*_i \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \Omega(\boldsymbol{\Phi}, H) && i = 1, \dots, N \\
\mathfrak{C}'_i &= \mathfrak{C}_i - \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*) \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \boldsymbol{\Phi}_i && i = 1, \dots, N \\
\boldsymbol{x}' &= \boldsymbol{x} - \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*) \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \Omega(\boldsymbol{\Phi}, H).
\end{aligned}$$

These are data for a new quadrilateral lattice $\boldsymbol{x}'$ as long as $\boldsymbol{\Phi}_i \in V$, V being a linear space and $\boldsymbol{\Phi}^*_i \in V^*$ being the dual of V, are solutions of (1)

$$\Delta_j \boldsymbol{\Phi}_i = (T_j Q_{ij}) \boldsymbol{\Phi}_j \tag{7}$$

and (2)

$$\Delta_j \boldsymbol{\Phi}^*_i = Q_{ji} T_j \boldsymbol{\Phi}^*_j \tag{8}$$

respectively. The linear operator $\Omega(\zeta, \xi^*) : W \to V$ is defined by the compatible equations:

$$\Delta_i \Omega(\zeta, \xi^*) = \zeta_i \otimes (T_i \xi_i^*) \qquad i = 1, \ldots, N. \tag{9}$$

When the data defining the fundamental transformation satisfy

$$\begin{aligned} \Phi_i &= (\Omega(\mathfrak{C}, \Phi^*) + T_i \Omega(\mathfrak{C}, \Phi^*))^\tau \mathfrak{C}_i \qquad i = 1, \ldots, N \\ \Omega(\Phi, \Phi^*) + \Omega(\Phi, \Phi^*)^\tau &= 2\Omega(\mathfrak{C}, \Phi^*)^\tau \Omega(\mathfrak{C}, \Phi^*) \end{aligned}$$

where $A^\tau := I_{p,q} A^{\mathrm{t}} I_{p,q}$, the transformation preserves the pseudo-circular reduction [4,10–12].

3. Transformation of the first potentials and of τ-functions

We now derive how the first potentials and τ-functions transform under a vectorial fundamental transformation.

In order to prove our first proposition, describing the transformation of the first potentials, we need the following two preliminary lemmas.

Lemma 1. *The functions* Φ_j^* *satisfy*

$$T_i T_j \Phi_i^* = \frac{1}{1 - (T_i Q_{ji})(T_j Q_{ij})} [(T_i Q_{ji})(T_j \Phi_j^*) + T_i \Phi_i^*]. \tag{10}$$

Proof. The linear system for Φ_j^*

$$\Delta_i \Phi_j^* = Q_{ij} T_i \Phi_i^*$$

can be written as

$$T_i \Phi_j^* = \Phi_j^* + Q_{ij} T_i \Phi_i^*.$$

By applying the T_j operator to this relation we obtain

$$T_j T_i \Phi_j^* = T_j \Phi_j^* + (T_j Q_{ij}) T_j T_i \Phi_i^* \tag{11}$$

so that, interchanging i and j

$$T_j T_i \Phi_i^* = T_i \Phi_i^* + (T_i Q_{ji}) T_j T_i \Phi_j^*$$

that when inserted in (11) gives

$$T_j T_i \Phi_j^* = T_j \Phi_j^* + (T_j Q_{ij})(T_i \Phi_i^* + (T_i Q_{ji}) T_j T_i \Phi_j^*).$$

Hence

$$(1 - (T_j Q_{ij})(T_i Q_{ji})) T_j T_i \Phi_j^* = T_j \Phi_j^* + (T_j Q_{ij}) T_i \Phi_i^*$$

and (10) follows. □

Lemma 2.

(1) For any vector $v \in V$

$$(T_j \Phi_j^*)(T_j \Omega(\Phi, \Phi^*)^{-1}) v = \frac{(T_j \Phi_j^*)\Omega(\Phi, \Phi^*)^{-1} v}{1 + (T_j \Phi_j^*)\Omega(\Phi, \Phi^*)^{-1} \Phi_j}. \tag{12}$$

(2) We have

$$\begin{aligned} T_j \Omega(\Phi, \Phi^*)^{-1} &= \Omega(\Phi, \Phi^*)^{-1} \\ &\quad - \frac{1}{1 + (T_j \Phi_j^*)\Omega(\Phi, \Phi^*)^{-1} \Phi_j} [\Omega(\Phi, \Phi^*)^{-1} \Phi_j] \otimes [(T_j \Phi_j^*)\Omega(\Phi, \Phi^*)^{-1}]. \end{aligned} \tag{13}$$

Proof.

(1) As we have $\Delta_j\Omega^{-1} = -\Omega^{-1}(\Delta_j\Omega)T_j\Omega^{-1}$, using (9) we deduce

$$T_j\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1} = \Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1} - \Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\boldsymbol{\Phi}_j\otimes(T_j\boldsymbol{\Phi}_j^*)T_j\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}. \tag{14}$$

Contracting (14) with $T_j\boldsymbol{\Phi}_j^*$ and v it follows that

$$\begin{aligned}(T_j\boldsymbol{\Phi}_j^*)T_j\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}v &= (T_j\boldsymbol{\Phi}_j^*)\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}v\\ &\quad -[(T_j\boldsymbol{\Phi}_j^*)\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\boldsymbol{\Phi}_j][(T_j\boldsymbol{\Phi}_j^*)(T_j\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1})v]\end{aligned}$$

so that

$$(1+(T_j\boldsymbol{\Phi}_j^*)\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\boldsymbol{\Phi}_j)(T_j\boldsymbol{\Phi}_j^*)T_j\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}v = (T_j\boldsymbol{\Phi}_j^*)\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}v$$

and (12) follows.

(2) We now contract (14) with v^* and v to obtain

$$v^*(T_j\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1})v = v^*\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}v - [v^*\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\boldsymbol{\Phi}_j][(T_j\boldsymbol{\Phi}_j^*)T_j\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}v].$$

Now using (12) we deduce (13).

□

We are now ready to describe the behaviour of the first potentials under vectorial fundamental transformations.

Proposition 1. *The first potentials transform according to*

$$\rho_i' = \rho_i(1+(T_i\boldsymbol{\Phi}_i^*)\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\boldsymbol{\Phi}_i). \tag{15}$$

Proof. For the proof of this proposition we introduce the following notation:

$$A_{ij} := 1-(T_jQ_{ij})(T_iQ_{ji}) \qquad \gamma_j := 1+a_{jj} \qquad a_{ij} := (T_i\boldsymbol{\Phi}_i^*)\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\boldsymbol{\Phi}_j.$$

On the one hand we have

$$T_j\gamma_i = 1+(T_iT_j\boldsymbol{\Phi}_i^*)(T_j\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1})T_j\boldsymbol{\Phi}_i,$$

from which, using (10), (14) and $\Delta_j\boldsymbol{\Phi}_i = (T_jQ_{ij})\boldsymbol{\Phi}_j$, we derive

$$\begin{aligned}T_j\gamma_i &= 1+\frac{1}{A_{ij}\gamma_j}\big(\big[(T_iQ_{ji})(T_j\boldsymbol{\Phi}_j^*)+T_i\boldsymbol{\Phi}_i^*\big]\\ &\quad\times\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\big[\gamma_j-\boldsymbol{\Phi}_j\otimes(T_j\boldsymbol{\Phi}_j^*)\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\big]\big[(T_jQ_{ij})\boldsymbol{\Phi}_j+\boldsymbol{\Phi}_i\big]\big).\end{aligned}$$

Expanding the above products we obtain

$$\begin{aligned}T_j\gamma_i &= 1+\frac{1}{A_{ij}\gamma_j}\big((T_iQ_{ji})(T_jQ_{ij})(\gamma_j-a_{jj})a_{jj}\\ &\quad+(T_jQ_{ij})(\gamma_j-a_{jj})a_{ij}+(T_iQ_{ji})(\gamma_j-a_{jj})a_{ji}+\gamma_ja_{ii}-a_{ij}a_{ji}\big).\end{aligned}$$

Now, as $\gamma_j - a_{jj} = 1$ we have

$$\begin{aligned}T_j\gamma_i &= \frac{1}{A_{ij}\gamma_j}\big((1-(T_iQ_{ji})(T_jQ_{ij}))(1+a_{jj})+(T_iQ_{ji})(T_jQ_{ij})a_{jj}\\ &\quad+(T_jQ_{ij})a_{ij}+(T_iQ_{ji})a_{ji}+(1+a_{jj})a_{ii}-a_{ij}a_{ji}\big).\end{aligned}$$

Then, we deduce

$$\begin{aligned}A_{ij}\frac{T_j\gamma_i}{\gamma_i} &= \frac{1}{\gamma_i\gamma_j}(1-(T_iQ_{ji})(T_jQ_{ij})+(T_jQ_{ij})a_{ij}+(T_iQ_{ji})a_{ji}\\ &\quad+a_{jj}+a_{ii}+a_{ii}a_{jj}-a_{ij}a_{ji}).\end{aligned}$$

On the other hand we have

$$T_j Q'_{ij} = T_j Q_{ij} - (T_j \mathbf{\Phi}^*_j)(T_j \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^{-1})(T_j \mathbf{\Phi}_i)$$
$$= T_j Q_{ij} - (T_j \mathbf{\Phi}^*_j)\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^{-1}((T_j Q_{ij})\mathbf{\Phi}_j + \mathbf{\Phi}_i)\frac{1}{\gamma_j}$$
$$= \frac{(T_j Q_{ij}) - a_{ji}}{\gamma_j}. \tag{16}$$

Thus,

$$1 - (T_j Q'_{ij})(T_i Q'_{ji}) = 1 - \frac{1}{\gamma_i \gamma_j}((T_j Q_{ij}) - a_{ji})((T_i Q_{ji}) - a_{ij})$$
$$= \frac{1}{\gamma_i \gamma_j}(1 + a_{ii} + a_{jj} + a_{ii}a_{jj} - (T_i Q_{ji})(T_j Q_{ij})$$
$$+ a_{ij}(T_j Q_{ij}) + a_{ji}(T_i Q_{ji}) - a_{ji}a_{ij})$$

and we conclude

$$1 - (T_j Q'_{ij})(T_i Q'_{ji}) = (1 - (T_j Q_{ij})(T_i Q_{ji}))\frac{T_j \gamma_i}{\gamma_i}.$$

Hence, recalling (3) we deduce

$$\frac{T_j \rho'_i}{\rho'_i} = 1 - (T_j Q'_{ij})(T_i Q'_{ji}) = \frac{T_j(\rho_i \gamma_i)}{\rho_i \gamma_i}$$

and therefore we can choose

$$\rho'_i = \rho_i \gamma_i.$$

□

We now show that the τ-functions transform according to a very simple rule.

Proposition 2. *The τ-function transforms according to*

$$\tau' = \tau \det \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*).$$

Proof. We first observe that

$$a_{ii} = \operatorname{tr}(\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^{-1} \Delta_i \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)).$$

Secondly, if we denote the rows of $\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)$ as $\mathbf{\Omega}_i$—which we consider now as elements of V^*—we know that

$$\det \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) = \operatorname{Det}(\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_d)$$

where $d = \dim V$, and Det is a skew multi-linear form on V^*. Thus, taking into account the discrete Leibnitz rule for the difference operator Δ_i we can write

$$\Delta_i \det \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) = \sum_{k=1}^{d} \operatorname{Det}(T_i \mathbf{\Omega}_1, \ldots, T_i \mathbf{\Omega}_{k-1}, \Delta_i \mathbf{\Omega}_k, \mathbf{\Omega}_{k+1}, \ldots, \mathbf{\Omega}_d).$$

But, as we have $T_i \mathbf{\Omega}_k = \mathbf{\Omega}_k + (\mathbf{\Phi}_i)_k T_i \mathbf{\Phi}^*_i$, where $(\mathbf{\Phi}_i)_k$ is the kth component of $\mathbf{\Phi}_i$, we have

$$\operatorname{Det}(T_i \mathbf{\Omega}_1, \ldots, T_i \mathbf{\Omega}_{k-1}, \Delta_i \mathbf{\Omega}_k, \mathbf{\Omega}_{k+1}, \ldots, \mathbf{\Omega}_d) = \operatorname{Det}(\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_{k-1}, \Delta_i \mathbf{\Omega}_k, \mathbf{\Omega}_{k+1}, \ldots, \mathbf{\Omega}_d).$$

Thus,

$$\Delta_i \det \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) = \sum_{k=1}^{d} \operatorname{Det}(\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_{k-1}, \Delta_i \mathbf{\Omega}_k, \mathbf{\Omega}_{k+1}, \ldots, \mathbf{\Omega}_d)$$

and using the Cramer rule we conclude

$$\frac{\Delta_i \det \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)}{\det \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)} = \operatorname{tr}(\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^{-1} \Delta_i \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)).$$

Therefore

$$\frac{T_i \det \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)}{\det \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)} = 1 + a_{ii}.$$

Then, recalling the previous proposition we deduce that

$$\frac{T_i \tau'}{\tau'} = \frac{T_i(\tau \det \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*))}{\tau \det \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)}$$

and the proposition follows. □

4. Vectorial fundamental transformations for symmetric lattices and pseudo-Egorov lattices

For a symmetric lattice we have:

Lemma 3. *Given a solution* $\mathbf{\Phi}^*$ *of* (8) *then a solution* $\mathbf{\Phi}$ *of* (7) *can be chosen so that*

$$A\rho_i (T_i \mathbf{\Phi}_i^*)^{\mathrm{t}} = \mathbf{\Phi}_i$$

with A an arbitrary linear operator over V. With this choice of the transformation it is consistent to take the potential $\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)$ *fulfilling the relation*

$$\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) A^{\mathrm{t}} = A \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^{\mathrm{t}}.$$

Proof. We have

$$\Delta_j \mathbf{\Phi}_i = A[(T_j \rho_i)(T_j T_i \mathbf{\Phi}_i^*) - \rho_i T_i \mathbf{\Phi}_i^*]^{\mathrm{t}}$$

now from (3) and (10) we deduce

$$\Delta_j \mathbf{\Phi}_i = A[\rho_i (T_i Q_{ji})(T_j \mathbf{\Phi}_j^*)]^{\mathrm{t}}$$

and using (5) we obtain (7) for $\mathbf{\Phi}_i$. Now, using (9), we compute

$$\Delta_i(\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) A^{\mathrm{t}} - A \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^{\mathrm{t}}) = A\rho_i (T_i \mathbf{\Phi}_i^*)^{\mathrm{t}} \otimes T_i \mathbf{\Phi}_i^* A^{\mathrm{t}} - A\rho_i (T_i \mathbf{\Phi}_i^*)^{\mathrm{t}} \otimes T_i \mathbf{\Phi}_i^* A^{\mathrm{t}} = 0$$

and the constraint

$$\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) A^{\mathrm{t}} = A \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^{\mathrm{t}}$$

holds for all points of the lattice whenever it is true at a single point of the lattice. □

The next proposition gives sufficient conditions on the transformation data to ensure that a vectorial fundamental transformation of a symmetric lattice gives a symmetric lattice.

Proposition 3. *Assume that the transformation data satisfy*

$$A\rho_i (T_i \mathbf{\Phi}_i^*)^{\mathrm{t}} = \mathbf{\Phi}_i \tag{17}$$

$$\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) A^{\mathrm{t}} = A \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^{\mathrm{t}}. \tag{18}$$

Then,

$$\rho'_j T_j Q'_{ij} = \rho'_i T_i Q'_{ji}.$$

Proof. From (16) we know that

$$\gamma_j T_j Q'_{ij} = (T_j Q_{ij}) - a_{ji}.$$

Thus, using proposition 1 we derive

$$\rho'_j T_j Q'_{ij} = \rho_j T_j Q_{ij} - \rho_j (T_j \boldsymbol{\Phi}^*_j) \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \boldsymbol{\Phi}_i$$

and from (17) we deduce

$$\rho'_j T_j Q'_{ij} = \rho_j T_j Q_{ij} - \boldsymbol{\Phi}^{\mathrm{t}}_j \Lambda \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \boldsymbol{\Phi}_i.$$

Thus,

$$\rho'_j (T_j Q'_{ij}) - \rho'_i (T_i Q'_{ji}) = \boldsymbol{\Phi}^{\mathrm{t}}_j [(\Lambda \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1})^{\mathrm{t}} - \Lambda \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1}] \boldsymbol{\Phi}_i$$

and the proposition follows from (18). □

Following [9] a pseudo-Egorov lattice is characterized by the identity

$$\rho^{\mathrm{S}}_i = \rho^{\mathrm{C}}_i$$

between the symmetric first potential ρ^{S}_i and the pseudo-circular one ρ^{C}_i. As the new potentials ρ'_i are related to the initial ones ρ_i by (15) then

$$\rho^{\mathrm{S}'}_i = \rho^{\mathrm{C}'}_i.$$

This implies that given a pseudo-Egorov lattice, a fundamental transformation over it preserving both symmetric and pseudo-circular character gives a new pseudo-Egorov lattice. Thus:

Proposition 4. *If the transformation data satisfies*

$$\begin{aligned}
\boldsymbol{\Phi}_i &= A \rho_i (T_i \boldsymbol{\Phi}^*_i)^{\mathrm{t}} \\
\boldsymbol{\Phi}_i &= (\Omega(\mathfrak{C}, \boldsymbol{\Phi}^*) + T_i \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*))^{\tau} \mathfrak{C}_i \\
\Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) A^{\mathrm{t}} &= A \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{\mathrm{t}} \\
\Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) &= \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*)^{\tau} \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*)
\end{aligned}$$

then the pseudo-Egorov reduction is preserved.

Acknowledgment

This work was partially supported by Dirección General de Enseñanza Superior e Investigación Científica proyecto PB98-0821.

References

[1] Bobenko A 1996 *Symmetries and Integrability of Difference Equations II* ed P Clarkson and F Nijhoff (Cambridge: Cambridge University Press)
Bobenko A and Pinkall U 1996 *J. Diff. Geom.* **43** 527
Bobenko A and Pinkall U 1996 *J. Reine Angew. Math.* **475** 187
Bobenko A and Schief W K 1998 *Discrete Integrable Geometry and Physics* ed A Bobenko and R Seiler (Oxford: Oxford University Press)
[2] Bogdanov L V and Konopelchenko B G 1995 *J. Phys. A: Math. Gen.* **28** L173–8
[3] Cieśliński J, Doliwa A and Santini P M 1997 *Phys. Lett.* A **235** 480
[4] Doliwa A 2000 *J. Geom. Phys.* **30** 169
[5] Doliwa A, Mañas M, Martínez Alonso L, Medina E and Santini P M 1999 *J. Phys. A: Math. Gen.* **32** 1197
[6] Doliwa A, Manakov S V and Santini P M 1998 *Commun. Math. Phys.* **196** 1
[7] Doliwa A, Santini P M and Mañas M 2000 *J. Math. Phys.* **41** 944
[8] Doliwa A and Santini P M 1997 *Phys. Lett.* A **233** 365
[9] Doliwa A and Santini P M 2000 *J. Geom. Phys.* **36** 60

[10] Konopelchenko B G and Schief W K 1998 *Proc. R. Soc.* A **454** 3075
[11] Liu Q P and Mañas M 1998 *Phys. Lett.* A **249** 424
[12] Mañas M *J. Math. Phys.* at press
[13] Mañas M, Doliwa A and Santini P M 1997 *Phys. Lett.* A **232** 365
[14] Martin R R 1986 *The Mathematics of Surfaces* ed J Gregory (Oxford: Clarendon)
Nutborne A W 1986 *The Mathematics of Surfaces* ed J Gregory (Oxford: Clarendon)
[15] Sauer R 1970 *Differenzengeometrie* (Berlin: Springer)